Solving Hard Sudoku

Sometimes applying the standard solving techniques to a difficult sudoku puzzle can lead you to an apparent impasse. No matter how hard you try, you can’t seem to overcome it. I’m going to show you an excellent way to get past any such obstacle, so long as the puzzle has a unique solution. It isn’t always pretty, but it always works.

To simplify the presentation, the first example shown here involves an impasse arrived at rather late in solving. Take a few moments to look over the diagram, and convince yourself that it won’t succumb to the usual array of solving techniques. It’s clear that the 14 blank squares must be filled with 2s, 5s, and 8s, but in what arrangement?

First notice that I’ve filled the remaining possibilities in the first diagram in gray and that several of the squares hold either a 2 or a 5. In the second diagram, I’ve labeled all of the (2,5) pairs as either A or B. That is, if one of them is A, another one in the same row, column, or region will be B. I’ve alternated between A and B as I worked around the puzzle. This is a handy way to show that the third square in the sixth row and the second square in the seventh row must hold the same number — they are both A. That is, they are both 5, or they are both 2. Suppose they were both 5. In that case, the first number in the fifth row would be an 8. Unfortunately, however, the first number in the ninth row would also then be an 8. In other words, we would have two 8s in the first column. This contraction means that our assumption was incorrect; both the numbers in question are 2s rather than 5s. From here, the puzzle solves normally again.

3	58	58	7	6	1	4	2	9
2	1	9	5	3	4	8	7	6
7	6	4	2	9	8	5	3	1
6	9	28	4	7	3	1	5	28
58	7	1	9	25	6	3	4	28
4	3	25	8	1	25	6	9	7
1	25	6	3	25	9	7	8	4
9	4	3	6	8	7	2	1	5
58	258	7	1	4	25	9	6	3

3	58	58	7	6	1	4	2	9
2	1	9	5	3	4	8	7	6
7	6	4	2	9	8	5	3	1
6	9	28	4	7	3	1	5	28
58	7	1	9	A	6	3	4	28
4	3	A	8	1	B	6	9	7
1	A	6	3	B	9	7	8	4
9	4	3	6	8	7	2	1	5
58	258	7	1	4	A	9	6	3

Briefly, all such solving obstacles can be removed by making a clever assumption and showing that it’s wrong! Before elucidating that point, let’s try a second, more complicated example.

Look at the incomplete puzzle in the diagram below. Take a moment to convince yourself that the last number in the sixth row is either a 5 or an 8. We’re going to get a contradiction by assuming it’s an 8. If I’d been working in pencil up to this point, I would now switch to pen and permanently set the squares I know for sure. Switching back to pencil, I write an 8 in the square in question.

17	9	17	5	4	2	8	6	3
5	4	2	3	8	6	1	7	9
3	8	6	7	9	1	5	4	2
9	2	5	6	7	48	3	18	148
6	3	8	149	15	459	7	2	145
17	17	4	28	25	3	6	9	58
4	6	19	189	3	89	2	5	7
28	17	137	1248	125	458	9	138	6
28	5	139	1289	6	7	4	138	18

Under my hypothesis,

1) The 4th number in the 6th row would be a 2, and the 8th number in the 4th row would be a 1.
2) Thus the 9th number in the 4th row would be a 4.
3) And the 6th number in the 4th row would be an 8.
4) And the 6th number in the 7th row would be a 9.
5) And the 8 in the 7th row would have to be in the 4th column.
6) And the 4th number in the 9th row would be a 1.

Aha! In that case, there would be no number that could legally occupy the ninth square in the ninth row since the 1 and 8 would already be taken. This contradiction proves that the hypothesis was wrong — the ninth number in the sixth row is a 5 and not an 8. Again, we can proceed normally to the end of the puzzle.

All right. Twice we’ve shown how choosing an appropriately false supposition can lead to the desired contradiction. How do you know what square to single out and which number to pick? You don't. These two examples worked out very tidily, but in practice this method can get quite messy. When you choose a supposition, three things can happen: 1) You can arrive at a contradiction, as we've seen; 2) You can hit another impasse without coming to a resolution either way; or 3) You can inadvertently choose the correct number to plug in and get all the way to a solution. In case 2, you're out of luck and must try another supposition (or leaving the assumption in place, make a further supposition). In case 3, you can either believe your constructor that there is exactly one solution (which you've found) and leave well enough alone, or you can set about to prove your answer correct by taking the opposite assumption and showing that it leads to a contradiction. Be prepared to do a lot of erasing.

Perhaps you can see why some people feel this method is not logical and/or involves guessing, and I have two responses to that feeling.
First, the idea here is simply to solve the puzzle. Why would one path to the solution be less valuable or appropriate than another? Some of us might have the voice of our algebra teacher in our ears — no partial credit for using your wits rather than algebraic logic. But this is fun and games, not math class.

Second, the method presented here is in fact completely logical and rigorous. What is called indirect proof (or sometimes proof by contradiction or, fancily, reductio ad absurdum) is fundamental to mathematics. Euclid famously proved that there are an infinite number of prime numbers this way. He supposed there were a finite number and showed that that led to an absurdity — just as we've done in our examples.

To be clear, there are other options available to solvers. If you search the Internet, you'll find a great number of direct-but-very-complicated techniques for pushing through the sorts of impasses discussed above. However, I always use the indirect-but-simple procedure I've just outlined. The idea is to make a hypothesis and test it, and the fact is we use hypotheticals like this constantly in solving sudoku (and elsewhere) — any time we eliminate a possibility using if-then reasoning. ("Could this be a 7? No, because then there'd be no 7 over here.") The difference here is that we can't hold all the if-thens in our head at the same time, and we have to resort to writing them down.

Here's one final example. Given the situation at the right, we'll make the supposition that the first number in the last row is a 6, and see if we can get a contradiction. In that case:

1) The 1st number in the 8th row would be 4.
2) The 1st number in the 6th row would be 8.
3) The 3rd number in the 4th row would be 6.
4) The 9th number in the 4th row would be 5.
5) The 9th number in the 7th row would be 6.
6) The 4th number in the 7th row would be 9.
7) The 4th number in the 9th row would be 8.

However, in that case, there would be no number left to go in the fourth position of the fourth row. This contradiction means that our original assumption was wrong; the first number in the last row is 3 rather than 6.

38	6	38	4	2	7	59	59	1
2	9	4	5	8	1	67	67	3
5	1	7	3	69	69	2	8	4
1	7	68	689	569	3	4	2	56
9	34	36	1	456	2	567	567	8
468	2	5	7	46	68	1	3	9
7	8	2	69	1	4	3	569	56
46	45	9	2	3	568	568	1	7
36	35	1	689	7	5689	5689	4	2